

SELF-CONSISTENT FIELD APPROACH TO TURBULENCE THEORY

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ABSTRACT

A self consistent field type perturbation theory is developed to treat the dynamics of stationary and homogeneous turbulence. The method consists in expanding the full probability distribution function about the product of exact single-mode distributions. The theory is used in second order to find expressions for the turbulent energy spectrum and associated response frequencies. The results for the energy spectrum are identical to a simplified form of the direct interaction approximation of Kraichnan, and closely resemble the results of the generalized random phase approximation of Edwards. The relation of the present method to both the above approaches is discussed.

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I. INTRODUCTION

The statistical theory of turbulence is usually treated in terms of the hierarchy of moment equations formed by multiplying the equation of motion by products of the velocity field and averaging the results over an ensemble of realizable flows. The equations for moments of different order are coupled together because of the nonlinearity in the equations of motion, and to obtain a closed set of equations it is necessary to make some statistical or dynamical assumption about the flow. An alternative approach, and one which has so far not been much used, is to study directly the full probability-distribution of the system. The probability-distribution function satisfies a linear equation, and the closure problem met with in the moment method is not explicitly encountered.

The approach described in the present paper is to reduce the equation for the full probability distribution to equations for single mode probability distributions by means of a self-consistent-field type calculation of the interaction amongst the modes. The procedure is similar to the Hartree self-consistent field calculation in quantum mechanics except that in the turbulence problem the analog of the single mode potential vanishes, so that the self-consistency requirement is of higher order. The perturbation expansion derived here is one in which the zeroth order term is a product of the exact single-mode distribution functions. Higher order terms contain the multi-mode correlations induced by the turbulent interactions, but contain explicitly no terms contributing to the product of single-mode distributions.

The method presented here is related to both the direct-interaction approximation of Kraichnan^{2,3} and the generalized-random-phase approximation of Edwards⁴. The basic ingredients of the direct interaction approximation are the wave number spectrum of kinetic energy and the time-correlation and response functions of the Fourier modes. The self-consistent field equations obtained here are identical with a simplified form of the direct interaction equations in which the time-correlation and response functions are approximated by exponential time dependence, and the response and correlation times for a given Fourier mode are assumed to be equal. The final equations then involve the energy-spectrum and the response time of the Fourier modes. Edwards proposes an expansion of the probability distribution based on a Fokker-Planck type characterization of the turbulence dynamics. The final equations also involve only the energy-spectrum and response times. They differ slightly from those of the present theory. The relationship between the present theory and that of Edwards is discussed in Section IV.

II. BASIC EQUATIONS

A. The Equations of Motion for Isotropic Turbulence

The Navier-Stokes equations for an incompressible viscous fluid are

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) \underline{v} = - \frac{1}{\rho} \nabla p - \underline{v} \cdot \nabla \underline{v} + F \quad (1a)$$

$$\nabla \cdot \underline{v} = 0$$

In (1a), ν is the kinematic viscosity, \underline{v} is the velocity field, ρ is the (constant) fluid density, p is the pressure field, and F is a possible solenoidal external driving force. We shall be interested in the stochastic properties of (1a), in the limit of an infinite medium, such that the pressure field becomes constant at spacial infinity.

It is convenient to rewrite (1a) in the following way. First introduce the Fourier transform of $\underline{v}(\underline{r}, t)$ defined by

$$\underline{v}(\underline{k}, t) = \int_{\underline{r}} \underline{v}(\underline{r}, t) e^{i \underline{k} \cdot \underline{r}}.$$

Then let $X_n(t)$ ($n=1, 2, \dots$) denote the real and imaginary part of all the $\underline{v}(\underline{k}, t)$. The index n labels both the wave number \underline{k} , and the vector indices of \underline{v} in any convenient fashion. The $X_n(t)$'s satisfy the following equation :

$$\left(\frac{d}{dt} + \nu_n \right) X_n = \sum_{pq} M_{npq} X_p X_q + F_n \quad (1b)$$

In equation (1b), the pressure field has been eliminated by using the incompressibility condition. The detailed specification of the M_{npq} is not needed for what follows. We note here a selection rule they satisfy assuring conservation of energy:

$$M_{npq} + M_{pnq} + M_{qnp} = 0 \quad (2)$$

We take M_{npq} to be symmetric in the last two indices, and put $M' = 0$ if any two indices are equal. This last condition means that the $k = 0$ amplitude of the velocity field vanishes. The driving force in (1) makes stationary turbulence possible against the presence of viscous drain. These forces will be taken to be negative dampings

$$F_n = + \nu'_n X_n.$$

B. Definition of the Probability Distribution Function

The probability distribution function, $P(X_1, X_2 \dots X_n \dots | t)$ is defined such that the probability that the phase of the system (1) will be found within a volume $(dX_1 \dots dX_n, \dots)$ around $(X_1, X_2, \dots X_n \dots)$ at time t is $P dX_1 dX_2 \dots$. It satisfies an equation of continuity which is

$$\left(\frac{\partial P}{\partial t} + \mathcal{L}_0 \right) P = VP, \quad (3)$$

where

$$\mathcal{L}_0 = - \sum_n \frac{\partial}{\partial X_n} (\nu_n X_n - F_n),$$

and

$$V = \sum_{npq} M_{npq} \frac{\partial}{\partial X_n} X_p X_q.$$

For a derivation of equation (3), see reference 1.

Some definitions will now be made which prove useful in what follows.

First define the one mode distributions by the equation

$$P_n(X_n) = \int d\Omega_{(n)} P(X_1, X_2 \dots X_n \dots) \quad (4)$$

where

$$d\Omega_{(n)} \equiv dX_1 \dots dX_{n-1} dX_{n+1} \dots$$

Next define the one mode projection of an operator $A(X_i, \partial/\partial X_j)$ as

$$\langle A \rangle = \sum \langle A \rangle_n \quad (5)$$

with

$$\langle A \rangle_n = \int d\Omega_{(n)} \prod_{j \neq n} P_j(X_j)$$

An operator A will be called a "one mode operator" if it satisfies the equation

$$\langle A \rangle = A \quad (6)$$

III. SELF CONSISTENT FIELD APPROXIMATION

A. Description of the Perturbation Method and Its Results

We shall be interested only in the statistically steady solutions of (3)

$$\mathcal{L}_0 P = VP. \quad (3a)$$

To gain an understanding of the method, consider first the case in which $V=0$. Then the solution for P is a product of single mode distribution P_n . If $V \neq 0$, P does not have the product form but can still be written as

$$P = \prod_n P_n + R, \quad (7)$$

where the P_n are the exact single mode distributions. The remainder term, R , contains all the multi-mode correlations induced by the turbulent interaction V . Note that the integral of R over $d\Omega_{(n)}$ must vanish, and consequently R makes no contribution to the single mode distributions. The perturbation method then consists in obtaining an expansion of R about $V = 0$.

To facilitate the quantitative development of such an expansion, it is useful to introduce an operator \mathcal{L} , having the following properties:

$$\mathcal{L} = \sum \mathcal{L}_n = \langle \mathcal{L} \rangle \quad (8a)$$

$$\int \mathcal{L}_n F(X_n) dX_n = 0, \quad (8b)$$

for any well-behaved F .

Equation (8a) asserts that \mathcal{L} is a single mode operator, and (8b) will assure the conservation of probability in what follows. In addition to \mathcal{L} it proves useful to introduce one mode functions $P_n^{\mathcal{L}}$ which satisfy

$$\mathcal{L} P^{\mathcal{L}} = 0, \quad (8c)$$

where $P^{\mathcal{L}} = \prod_n P_n^{\mathcal{L}}$.

Note that the properties prescribed by (8) for \mathcal{L} are also possessed by \mathcal{L}_0 . Hence it is consistent to require that $\mathcal{L} \rightarrow \mathcal{L}_0$ and $P^{\mathcal{L}} \rightarrow P$ as $V \rightarrow 0$.

Now introduce \mathcal{L} into (3a) to obtain

$$\mathcal{L}P = (\mathcal{L} - \mathcal{L}_0 + V)P. \quad (9)$$

The formal solution to (9) may be written as

$$P = P^{\mathcal{L}} + R^{\mathcal{L}}. \quad (10)$$

The first term in (10) is the solution to the homogeneous system (8c).

The term $R^{\mathcal{L}}$ is evaluated in section (IIIb). It depends on \mathcal{L} , \mathcal{L}_0 and V as well as the $P_n^{\mathcal{L}}$.

The program for the perturbation method may now be given more quantitative form. First, find an operator \mathcal{L} which satisfies conditions (8) such that $P_n^{\mathcal{L}} \equiv P_n$. Once this has been done, the second term in (10), $R^{\mathcal{L}}$, is identical to the remainder R in (7), so that R is now known in terms of \mathcal{L} and V . The final step, yielding the perturbation series is to expand R about $V = 0$.

The results of the perturbation method for \mathcal{L} and P are as follows:

$$\begin{aligned} \mathcal{L} = \mathcal{L}_0 &- \langle V \frac{1}{\mathcal{L}} V \rangle - \langle V \frac{1}{\mathcal{L}} V \frac{1}{\mathcal{L}} V \frac{1}{\mathcal{L}} V \rangle \\ &+ \langle V \frac{1}{\mathcal{L}} \langle V \frac{1}{\mathcal{L}} V \rangle \frac{1}{\mathcal{L}} V \rangle + \langle V \frac{1}{\mathcal{L}} V \frac{1}{\mathcal{L}} \langle V \frac{1}{\mathcal{L}} V \rangle \rangle \\ &\dots \end{aligned}$$

(11a)

$$\mathcal{L} P^{\mathcal{L}} = 0$$

(11b)

$$\begin{aligned} P = P^{\mathcal{L}} &+ \frac{1}{\mathcal{L}} V P^{\mathcal{L}} + \frac{1}{\mathcal{L}} \left\{ V \frac{1}{\mathcal{L}} V - \langle V \frac{1}{\mathcal{L}} V \rangle \right\} P^{\mathcal{L}} \\ &+ \frac{1}{\mathcal{L}} \left\{ V \frac{1}{\mathcal{L}} V \frac{1}{\mathcal{L}} V - V \frac{1}{\mathcal{L}} \langle V \frac{1}{\mathcal{L}} V \rangle - \langle V \frac{1}{\mathcal{L}} V \rangle \frac{1}{\mathcal{L}} V \right\} P^{\mathcal{L}} \\ &+ \dots \end{aligned}$$

(11c)

The operator $1/\hat{\mathcal{L}}$ in (11a) and (11c) is the inverse to the non-singular part of \mathcal{L} . Its definition is given by (14).

Equations (11a) and (11b) constitute a pair of equations for the joint determination of \mathcal{L} and P_n . The one-mode projections in (11a) are integrals over P_n so that (11a) and (11b) are coupled together in a self-consistent manner. Once \mathcal{L}_n and P_n are determined, P is to be found from (11c). With regard to (11c), note that terms of order higher than $P^{\mathcal{L}}$ make no contribution to the single mode P_n . On taking the one more average, terms odd in V vanish by symmetry, while the terms of any even order in V cancel amongst themselves.

The term "order of perturbation" is used here to denote the explicit number of V -operators occurring on the right hand side of (11a) and (11b). Truncation of the perturbation series at any finite order actually retains infinite powers of V because of the nonlinearity of (11a).

The equations for the intensities $\phi_n \equiv \int X_n^2 P_n dX_n$ may be obtained from equations (11a) and (11b). To second order these are

$$(\nu_n - \nu'_n) \phi_n = 2 \sum M^2_{npq} \frac{\phi_p \phi_q}{\eta_{n+p+q}} \quad (12a)$$

$$+ 4 \sum M_{npq} M_{pnq} \frac{\phi_n \phi_q}{\eta_{n+p+q}} ,$$

where

$$\eta_n = \nu_n - \nu'_n - 4 \sum M_{npq} M_{pnq} \frac{\phi_q}{\eta_{p+q}} . \quad (12b)$$

Here τ_n may be interpreted as a relaxation frequency of the mode n . It measures the rate of decay, due to the joint action of viscosity and turbulence, of a small perturbation of a mode amplitude. Thus, $1/\tau_{npq}$ and $1/\tau_{pq}$ are effective relaxation times for dynamically induced correlations to disappear.

The derivations of equations (11) and (12) follow in section (IIIb) and (IIIc) respectively. The relation of the present method to those of Kraichnan^{2,3} and Edwards⁴ are discussed in section IV, which may be read without going through the derivation.

B. Derivation of the Perturbation Method.

Our immediate goal is to find an operator \mathcal{L} satisfying (8) such that $P_n^{\mathcal{L}} = P_n$. To this end, rewrite (9) as

$$P = P^{\mathcal{L}} + \frac{1}{\mathcal{L}} (\mathcal{L} - \mathcal{L}_0 + V) P, \quad (13)$$

where

$$P^{\mathcal{L}} \equiv \prod_j P_j^{\mathcal{L}}(x_j).$$

Equation (13) is an integral equation for P . The first term is the solution to the homogeneous equation (8c). The second term needs some comment since the Green's operator $1/\mathcal{L}$ has not yet been defined. Some caution is necessary in its definition in view of the fact that \mathcal{L} has zero as one of its eigenvalues as is implied by (8c).

Let the eigenvalues of \mathcal{L} be denoted by λ_i , and the corresponding eigenfunctions by ψ_i . Note that (8c) implies that one of the λ_i , λ_1 , say, is zero and that the associate eigenfunction is $P^{\mathcal{L}}$. The eigenfunction ψ_i , together with the adjoint function $\tilde{\psi}_i$ are assumed to form biorthonormal set in terms of which any integrable function $f(x_1 \dots x_n, \dots)$ may be expanded:

$$f(X_1 X_2 \dots X_n \dots) = \sum_1^{\infty} C_n \psi_n (X_1 \dots X_n \dots)$$

where

$$C_n = (\tilde{\psi}_n, f) \equiv \int d\Omega f \tilde{\psi}_n .$$

Here $\tilde{\psi}_n$ are eigenfunctions of the operator adjoint to \mathcal{L} . Now define

$\hat{\mathcal{L}}^{-1}$ by the equation

$$\hat{\mathcal{L}}^{-1} f = \sum_{n \neq 1} \frac{1}{\lambda_n} (\tilde{\psi}_n, f) \psi_n, \quad (14)$$

for any function f . Using equation (14), it may be verified directly that P as given by (13) satisfies (9) provided that

$$(\tilde{\psi}_1, (\mathcal{L} - \mathcal{L}_0 + V) P) = 0,$$

which is a consistency condition that (9) has a nontrivial solution in the form specified by (13). In the next section it is shown that \mathcal{L}_n has the general form

$$\mathcal{L}_n = \frac{\partial}{\partial X_n} \Lambda_n,$$

where Λ_n is a nonsingular operator (see equation (20)). From this, one may show that $\tilde{P}_f = 1$, and hence the consistency condition is fulfilled.

We now return to the construction of an equation for \mathcal{L} . First, it is convenient to solve (13) formally by iterating the right hand side. There results

$$P = P_f + \frac{1}{\mathcal{L}} T P_f, \quad (15)$$

where

$$T = (\mathcal{L} - \mathcal{L}_0 + V) + (\mathcal{L} - \mathcal{L}_0 + V) \frac{1}{\mathcal{L}} T.$$

Now require that P_n^f be equal to P_n . From (15), it follows that a sufficient condition that this be so is

$$\left\langle \frac{1}{\mathcal{L}} T \right\rangle P_n^f = \frac{1}{\mathcal{L}} \langle T \rangle_n^f P_n^f = 0. \quad (16)$$

Here, the superscripts f specify that the one mode projectors are with respect to the function P^f . The first equality in (16) is obtained by expanding the denominator in the following Taylor Series:

$$\langle \frac{1}{\mathcal{L}} T \rangle_n = \frac{1}{\mathcal{L}_n} \langle T \rangle_n + \frac{1}{\mathcal{L}_n} \sum_{S=1}^{\infty} \left(\frac{1}{\mathcal{L}_n} \right)^S \left\langle \left(\sum_{q \neq n}^{\infty} \mathcal{L}_q \right)^S T \right\rangle_n.$$

The second term here vanishes, by (8b) since each term contains factors of the form

$$\int \mathcal{L}_q F_q dX_q = 0.$$

Equation (16) may be rewritten in a more convenient form by using (15), and the fact that $P_n = P_n^{\mathcal{L}}$. The result is

$$\left\{ \mathcal{L}_n - \mathcal{L}_n^0 + \left\langle V \frac{1}{\mathcal{L}} T \right\rangle_n \right\} P_n^{\mathcal{L}} = 0. \quad (17)$$

The operator \mathcal{L}_n is to be chosen consistent with (8a), (8b), (8c) and (17). This is achieved by requiring the operator in (17) to vanish, and keeping equation (8c) as an independent constraint:

$$\mathcal{L}_n = \mathcal{L}_n^0 - \left\langle V \frac{1}{\mathcal{L}} T \right\rangle_n \quad (18)$$

$$\mathcal{L}_n P_{\mathcal{L}} = 0.$$

The superscript \mathcal{L} has been dropped in (18) as it is now superfluous. The above choice of \mathcal{L}_n automatically satisfies (8c) since (18) is of the form

$$\mathcal{L}_n = \frac{\partial}{\partial X_n} \Lambda_n,$$

as may be verified from the definitions of \mathcal{L}_0 , V , and the single mode projector $\langle \rangle_n$.

The perturbation series for $\mathcal{L} - \mathcal{L}_0$, in terms of $V \frac{1}{\mathcal{L}} V$ may be found in the following way. Introduce the formal solution of T into (18) to obtain

$$\Delta = - \langle V \frac{1}{\mathcal{L}} \frac{1}{1 - (\Delta + V) \frac{1}{\mathcal{L}}} (\Delta + V) \rangle,$$

where

$$\Delta = \mathcal{L} - \mathcal{L}_0.$$

Solve this equation for Δ by iteration, using $\Delta = 0$ to initialize the right hand side. Then expand the denominator and collect equal powers of $V \frac{1}{\mathcal{L}} V$. The result is (11a). Finally the perturbation series for P (equation 11c) is obtained by introducing (11a) into (15), expanding T and coalescing equal powers of V .

To assess the meaning of the perturbation series, first note that by requiring $\langle T \rangle$ to vanish, an expansion has been achieved in which the zeroth order term is a product of the exact single mode distribution functions. Higher order terms represent progressively more complicated multimode correlations. The contribution of any order ($N > 0$) to the single mode P_n vanishes independent of any other order of perturbation.

We remark here, parenthetically, that equations (12) are analogous to formulas for the single particle potential in the quantum many body theory. The term $\langle V \hat{I}^{-1} T \rangle$ is the analogue of the optical potential^{5/}.

C. Derivation of the Second Order Perturbation Solution for the Intensities and Relaxation Frequencies.

To second order \mathcal{L}_n is given by

$$\mathcal{L}_n = \mathcal{L}_n^0 - \langle V \frac{1}{\mathcal{L}} V \rangle_n \quad (19)$$

This procedure extends the approximation method to the next higher order beyond the Hartree self-consistency requirement. Such a procedure assumes, according to (11c) that deviations of P from $P^{\mathcal{L}}$ are adequately represented by terms linear in the operator V .

Equation (19), may be expanded as

$$\mathcal{L}_n = \mathcal{L}_n^0 - 2 \sum_{p,q} M_{npq}^2 \frac{\partial}{\partial X_n} \int X_p X_q \frac{dX_p dX_q}{\mathcal{L}_n + \mathcal{L}_p + \mathcal{L}_q} X_p X_q P_p P_q \frac{\partial}{\partial X_n} \quad (20)$$

$$- 4 \sum_{p,q} M_{npq} M_{pnq} \frac{\partial}{\partial X_n} \int X_p X_q \frac{dX_p dX_q}{\mathcal{L}_n + \mathcal{L}_p + \mathcal{L}_q} \frac{\partial}{\partial X_p} X_q X_n P_p P_q$$

In deriving (20), (8b) has been used to reduce the denominators from sums over all modes to sums over the triad n, p and q . Also the symmetry condition on M has also been used. An equation for the intensities ϕ_n may now be obtained from the condition

$$\int X_n^2 \mathcal{L}_n P_n = 0.$$

Using (20) there results after some partial integrations

$$(v_n - v'_n) \phi_n = \sum_{p,q} \frac{2}{\eta_{npq}} M_{npq}^2 \phi_p \phi_q$$

(21a)

$$+ 4 \sum_{p,q} \frac{1}{\eta_{npq}} M_{npq} M_{pnq} \phi_n \phi_q$$

where

$$\frac{1}{\eta_{npq}} \equiv \frac{1}{\phi_p \phi_q} \iiint X_n X_p X_q \frac{\frac{dX_n}{\mathcal{L}_n + \mathcal{L}_p + \mathcal{L}_q} \frac{dX_p}{\mathcal{L}_n + \mathcal{L}_p + \mathcal{L}_q} \frac{dX_q}{\mathcal{L}_n + \mathcal{L}_p + \mathcal{L}_q}}{\frac{\partial}{\partial X_n} X_p X_q P_p P_q} . \quad (21b)$$

The η_{npq} are relaxation frequencies describing the joint relaxation of the modes n , p and q by the action of turbulence and viscosity. These must be evaluated to make the determination of ϕ_n complete. To this end consider the integral

$$I_n = \int X_n \mathcal{L}_n F_n(X_n) dX_n,$$

where F is an arbitrary function of X_n which vanishes sufficiently rapidly to make I_n finite. Using (20) there results

$$\int X_n \mathcal{L}_n F_n(X_n) dX_n = - \eta_n \int X_n F(X_n) dX_n \quad (22)$$

where
$$\eta_n = v_n - v'_n - 4 \sum M_{npq} M_{pnq} \frac{\phi_q}{\mathcal{L}_{pq}}$$

and
$$\eta_{pq} \equiv \frac{1}{\phi_q} \int \int \int X_p X_q \frac{dX_p dX_q}{\mathcal{L}_p + \mathcal{L}_q} \frac{\partial}{\partial X_n} X_q^p X_p^p .$$

Equation (22) is obtained from (20) by partial integration, and by noting that, for any well behaved function $G_n(X_n)$

$$\int dX_n \frac{1}{\mathcal{L}_n + \mathcal{L}_p + \mathcal{L}_q} G_n(X_n) = \frac{1}{\mathcal{L}_p + \mathcal{L}_q} \left(\int G_n(X_n) dX_n \right) . \quad (23)$$

since
$$\int dX_n \mathcal{L}_n^r G_n(X_n) = 0 \text{ if } r \neq 0, \text{ from (8c).}$$

Now put

$$F(X_n) = \mathcal{L}_n^p G_n(X_n)$$

into (22) and define

$$Q_n^{(P)} \equiv \int X_n f_n^P G_n dX_n \quad .$$

There results

$$Q_n^{(P+1)} = - \eta_n Q_n^{(P)} \quad (24)$$

or

$$Q_n^P = - (-\eta_n)^P Q_n^{(0)}.$$

Using (24), η_{pq} and η_{npq} may now be evaluated by expanding the denominator in Taylor series. There results

$$\frac{1}{\eta_{pq}} = \frac{1}{\phi_q} \int X_p X_q \frac{dX_p dX_q}{f_p + f_q} \frac{\partial}{\partial X_p} X_q^P \frac{\partial}{\partial X_q} X_q^P$$

$$= \frac{1}{\phi_q} \int X_p X_q \frac{dX_p}{f_p} \sum_{s=0}^{\infty} \int \left(\frac{-f_q}{f_p} \right)^s X_q^P dX_q \frac{\partial^2}{\partial X_p^2}$$

$$= \int X_p \frac{1}{f_p - \eta_q} \frac{\partial^2}{\partial X_p^2} dX_p \quad .$$

After some further steps there results

$$\eta_{pq} = \eta_p + \eta_q,$$

and similarly

$$\tau_{hpq} = \tau_h + \tau_p + \tau_q .$$

Using the above results for the τ 's in (21) gives (12a) and (12b) for φ_n and τ_n .

IV. COMPARISON OF PRESENT METHOD TO THAT OF EDWARDS

The present work is related to the direct interaction approximation of Kraichnan^{2, 3}, and the generalized random phase approximation of Edwards⁴. As stated in the introduction, our results are identical to a simplification of the direct interaction equations in which the response and correlation functions are approximated by exponential functions and the response and correlation times are assumed to be equal. The generalized random phase approximation gives results identical to (12), except that the denominator $\eta_p + \eta_q$ (in 12b) is replaced by $\eta_n + \eta_p + \eta_q$. It is of interest, to seek the reason for this difference. We first give a brief outline of Edward's approach.

Edwards assumes that the single mode equation is well approximated by a Fokker-Planck equation

$$\mathcal{L}_n^{(F)} P_n^{(F)} = 0 \quad (25)$$

where

$$\mathcal{L}_n^{(F)} = - \frac{\partial}{\partial X_n} \left(\eta_n X_n + K_n \frac{\partial}{\partial X_n} \right),$$

$$P_n^{(F)} = \frac{1}{\sqrt{2\pi\phi}} e^{-X_n^2/2\phi_n},$$

and

$$\frac{K_n}{\eta_n} = \phi_n.$$

Here, η_n is a dynamical friction coefficient and K_n an eddy diffusion coefficient. Their evaluation is made through an examination of the exact form of the full probability distribution P . For the latter, Edwards introduces a perturbation expansion. The equation for P is written in the form

$$(\mathcal{L}^{(F)} - \lambda V + \lambda^2 (\mathcal{L}_0 - \mathcal{L}^{(F)}) P = 0.$$

Here, λ is an ordering parameter which is put equal to unity at the end of the calculation. A solution for P is sought in the form

$$P = \sum_{r=0}^{\infty} \lambda^r P^{(r)},$$

and the $P^{(r)}$ are found to be

$$P^{(0)} = \prod_j P_j^{(F)}(x_j) \equiv P^{(F)}$$

$$P^{(1)} = \frac{1}{\mathcal{L}^{(F)}} V P^{(F)}$$

$$P^{(2)} = \frac{1}{L_F} \left\{ v \frac{1}{L_F} v - (L - L_F) \right\} P^{(F)}$$

(26)

The intensity normalization on P_n is

$$\int X_n^2 P_n dX_n = \phi_n$$

and the same normalization is imposed by Edwards on $P_n^{(F)}$. Therefore, contributions to ϕ_n from $P^{(r)}$ for $r > 0$ must vanish. Thus for any order of approximation N it is required that

$$\sum_{r=1}^N \int X_n^2 P_n^{(r)} dX_n = 0 \quad (27)$$

The N^{th} order approximations for K_n and η_n are to be chosen so as to conform to (27). Edwards obtains explicit results only for the second order approximation and the remainder of the present discussion is confined to this order. One more condition in addition to (27) is necessary to make determinate equations. The additional condition is imposed on K_n . It is required that K_n have the form which would be appropriate for describing the action of the turbulent force $\sum M_{npq} X_p X_q$ on the mode X_n if this force were completely Gaussian.

To second order η_n and K_n are chosen according to

$$\left\{ \mathcal{L}_n^0 - \mathcal{L}_n^F - \left\langle v \frac{1}{\mathcal{L}_F} v \right\rangle^F \right\} P_n^{(F)} = 0, \quad (28a)$$

and

$$K_n = \sum \frac{M_{npq}^2 \phi_p \phi_q}{\eta_n + \eta_p + \eta_q} \quad (28b)$$

Equation (28a) is very similar to (19); the self-consistent field method requires the operator itself to vanish, while (28a) requires the product of the operator with P_n to vanish. Conditions (28) lead to a choice of η_n and $\phi_n = K_n/\eta_n$ very close to (12a) and (12b). The only difference is that if (28) is used to obtain the intensities ϕ_n then the denominator $\eta_p + \eta_q$ in (12b) is replaced by $\eta_n + \eta_p + \eta_q$.

We are now in a position to make a comparative assessment of the two methods. First with regard to the generalized random phase approximation, it is to be noted that the choice of η_n and K_n prescribed by (28) is not unique. In principle there are any number of K_n 's and η_n 's satisfying (27) which may be used. The choice given by (28) is based on the idea that the turbulent force is close to Gaussian, and that the Fokker-Planck equation is well suited to the description of the stochastic behavior of the amplitudes X_n . On the other hand, the method

proposed in the present paper is unique, in the sense that no dynamical or statistical assumptions are necessary in determining the form of the operator \mathcal{L} . The latter is entirely determined by the condition that the zeroth order term in the perturbation series be the product exact single mode distribution. The form of \mathcal{L} is not that of a Fokker-Planck operator, as may be verified from (20).

Another difference in the two methods appears in the roles played by the one mode distribution P_n in the self-consistent field method, and $P_n^{(F)}$ in the generalized random phase method. In the latter, the $P_n^{(F)}$ serve as approximations to the exact single mode distributions P_n . The exact P_n have terms contained in higher orders of the perturbation series (26). On the other hand, the present method employs the exact single mode distributions as the leading term in the perturbation series and, consequently, no higher order terms contribute to it.

In this connection, note also that in the generalized random phase approximation the $P_n^{(F)}$ are strictly Gaussian, whereas in the present method the P_n are to be computed from the self-consistent set of equations, (11a) and (11b). In deriving the second order results for ϕ_n and η_n it was not necessary to make any explicit use of the form of P_n , except for continuity and symmetry properties. The calculation of the P_n by the self-consistent field method, even to second order, seems to be a rather difficult task. However, in the case in which the turbulence is spatially homogeneous such a calculation is unnecessary, because in this case one may argue that P_n is Gaussian. The Gaussian form of P_n is a consequence of the fact that for homogeneous turbulence the velocity fields in distant regions of space are statistically independent.

The Fourier coefficients of the velocity field are effectively a sum of independent random variables and, by the central limit theorem have Gaussian univariate distributions. The present method consistently allows the assumption of Gaussian form to be used in evaluating \mathcal{L} to any finite order, and in a subsequent calculation a determination of the energy spectrum as described in section III to be carried out.

The solution for P_n has not been fully investigated. An interesting question here is whether in the case of homogeneous turbulence the Gaussian form is preserved to finite orders of perturbation theory.

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